

Infinite heat capacity anomaly for two-level systems with a mean-field coupling

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This paper examines the statistical mechanics of collections of two-level systems that experience a nonlinear mean-field coupling. The coupling reduces the two-level excitation energy by an amount that depends on the number of excitations already present; in other words the energy spectrum manifests a cooperative "softening" phenomenon. An exact solution in the infinite system limit is presented for key aspects of a specific case, the "logarithmic" model. This model exhibits a symmetric infinite heat capacity anomaly with divergence exponent equal to $2/3$.

Furthermore, the divergence evolves from maxima with heights asymptotically proportional to $N^{1/2}$ as the number N of coupled two-level systems increases to infinity.

I. INTRODUCTION

One of the basic objectives of statistical mechanics is to show how strong interactions among many degrees of freedom produce collective phenomena. The study of phase transitions offer the most obvious example, with critical phenomena having supplied a particularly intense and productive focus for research over the last two decades.¹⁻³

This paper explores the properties of a class of statistical mechanical models which exhibit phase transitions. In spite of the fact that they are conceptually simple, exactly solvable, and nontrivial in their properties, they apparently have not been examined previously. These models deserve consideration because they can exhibit critical behavior with an infinite-heat capacity anomaly. The analysis offered below shows that this property arises collectively from a special form of "mean-field" coupling between the degrees of freedom. That coupling enters in an unconventional nonlinear way which qualitatively alters the usual mean-field classical critical behavior with a bounded heat capacity.^{4,5}

Section II defines the new class of models, and displays their generic energy spectrum in such a way that the opportunity for phase transitions becomes obvious. In the interests of subsequent explicit calculations, a specific case from this general class of models is selected for special scrutiny, called below the "logarithmic model."

In general, both first-order and higher-order phase transitions can arise. However, Sec. III demonstrates for the specific case of the logarithmic model how to identify that value of a basic coupling constant which ensures that a critical point occurs at vanishing external field.

Section IV derives thermodynamic critical exponents for the logarithmic model. As its infinite heat capacity anomaly indicates, some of these at first sight appear to differ from those of conventional mean-field theory.

Section V examines the way that collective critical behavior emerges as the size of the system (i.e., the number of degrees of freedom N) increases to infinity. In particular, the analysis shows that the heat capacity (per degree of freedom) possesses a maximum which diverges to infinity as $N^{1/2}$. That characteristic is illustrated by some numerical results for fixed small values of N .

This paper concludes with a discussion in Sec. VI of the

relationships between the present models and those of the Ising type. In particular it is stressed that the peculiar critical exponents arise effectively from exchange of variables present in the usual mean-field equation of state.

II. ENERGY SPECTRUM

The models under consideration comprise N identical degrees of freedom, each of which is described by a state variable n_j . These latter are permitted to adopt only the values 0 and 1, corresponding respectively to ground and excited states. Units will be chosen so that the excitation energy is 1 for any one of these two-level degrees of freedom if it is isolated.

Precise definition of the models requires that the full set of 2^N energy levels $E(n_1, \dots, n_N)$ be specified. Because the component two-level systems are identical, E must have full permutational symmetry under interchange of the n_j . We now postulate that the energy spectrum has the following generic form:

$$E(n_1, \dots, n_N) = \sum_{j=1}^N n_j [\phi(M/N) + H]. \quad (2.1)$$

Here H is an external field acting equally on all two-level systems, and

$$M = \sum_{i=1}^N n_i \quad (2.2)$$

gives the total number of excitations present. The function $\phi(x)$ is required to be analytic on the unit interval $0 < x < 1$ and to satisfy the following conditions on that interval:

$$\phi(0) = 1, \quad (2.3a)$$

$$\phi(x) > 0, \quad (2.3b)$$

$$\phi'(x) < 0. \quad (2.3c)$$

Consequently ϕ is monotonically decreasing. When $H \geq 0$ the ground state will have all $n_j = 0$, but if

$$H < -\phi(1), \quad (2.4)$$

the ground state will have all $n_j = 1$.

The function ϕ mediates interactions between the two-level systems, and because it does so only through the variable M/N it provides a mean-field coupling that acts uni-

formly over the entire collection of two-level systems. The energy required to increase n_1 from 0 to 1 is

$$E(1, n_2, n_3, \dots) - E(0, n_2, n_3, \dots) = \phi(M/N) + H, \quad (2.5)$$

where M refers to the state with $n_1 = 1$. In the large system limit, this energy difference reduces to that for an isolated two-level system subject to the external field, provided n_2, n_3, \dots all are equal to zero. However, as the fraction of these other systems in the upper state increases from zero, ϕ decreases and so too will the excitation energy for system 1 decrease.

This is the qualitative basis which permits phase transitions in the models under study. Suppose for the moment that $H = 0$. Thermal equilibrium at low temperature would entail very few $n_j = 1$, and each two-level system would thereby experience virtually the full energy gap for excitation. As temperature rises, the fraction of systems in the upper state will also rise, and because that causes ϕ and the gap to decrease a positive feedback mechanism exists to enhance the fraction of excitations even further. If ϕ declines sufficiently rapidly with its variable M/N then an avalanche of excitations should ensue, in other words a phase transition.

The canonical partition function $Z(\beta, H)$, $\beta = 1/k_B T$, can be written compactly as a sum over M :

$$Z(\beta, H) = \sum_{M=0}^N [N!/M!(N-M)!] \times \exp\{-\beta M[\phi(M/N) + H]\}. \quad (2.6)$$

This is the generating function from which equilibrium thermodynamic properties can be deduced for any system size N . In the large system limit ($N \rightarrow \infty$) Z will be asymptotically dominated by terms in the M sum narrowly distributed around that value $M^*(\beta, H)$ which maximizes the summand. On account of the identity of all N two-level systems, this is given by $N \langle n \rangle$, where $\langle n \rangle$ is the thermal average value of any of the n_j . It is straightforward to show that

$$\langle n \rangle = \{1 + \exp[\beta d(\langle n \rangle \phi(\langle n \rangle))/d \langle n \rangle + \beta H]\}^{-1}. \quad (2.7)$$

For the purposes of specific detailed illustration below it will be useful to examine the following choice for ϕ (the logarithmic model):

$$\phi(x) = \ln(1 + \lambda x)/\lambda x. \quad (2.8)$$

The coupling constant λ is real and positive, and will be set at a value derived in Sec. III. The conditions (2.3) and the requirement of analyticity over $0 \leq x < 1$ are obviously obeyed. With use of this logarithmic form for ϕ the partition function becomes

$$Z(\beta, H) = \sum_{M=0}^N [N!/M!(N-M)!] \times (1 + \lambda M/N)^{-\beta N/\lambda} \exp(-\beta M H), \quad (2.9)$$

while the implicit equation (2.7) for $\langle n \rangle$ reduces to

$$\langle n \rangle = \{1 + \exp[\beta/(1 + \lambda \langle n \rangle) + \beta H]\}^{-1}. \quad (2.10)$$

This last expression makes it clear how λ controls the extent to which the ambient mean excitation field reduces the energy gap for excitation.

III. CRITICAL COUPLING CONSTANT

The next objective is identification of a value for the coupling constant λ such that when $H = 0$ the logarithmic model (in the $N \rightarrow \infty$ limit) manifests a second-order phase transition. This critical coupling constant λ_c marks the boundary at $H = 0$ between those small λ 's for which the behavior is characteristic of nearly independent degrees of freedom with no phase transition, and those large λ 's which induce first-order phase change with a jump discontinuity in $\langle n \rangle$ vs β .

With vanishing external field, Eq. (2.10) which determines $\langle n \rangle$ may be written

$$\langle n \rangle = f(\lambda \langle n \rangle, \beta), \quad (3.1)$$

where

$$f(x, \beta) = \{1 + \exp[\beta/(1 + x)]\}^{-1}. \quad (3.2)$$

That two coupling regimes are possible is illustrated by Fig. 1, where for $\beta = 6$ and two distinctly different λ choices, both members of Eq. (3.1) are plotted as a function of $\langle n \rangle$. Weak coupling ($\lambda = 5$ in Fig. 1) involves only a single intersection between the f curve and the 45° straight line, and this applies at all temperatures. Strong coupling ($\lambda = 20$ in Fig. 1) can yield three intersections over some nonvanishing β range, the relevant one of which (giving the absolute summand maximum in Z) switches its identity at the transition temperature.

The desired critical point corresponds to confluence of the three intersections. When this occurs the f curve is tangent to the 45° line at the intersection, and has a point of inflection there as well. The simultaneous values $\langle n \rangle_c$, β_c , and λ_c which locate the critical point therefore must be determined from the conditions

$$f(\lambda_c \langle n \rangle_c, \beta_c) = \langle n \rangle_c, \quad (3.3)$$

$$[\partial f / \partial \langle n \rangle]_{\lambda_c \langle n \rangle_c, \beta_c} = 1, \quad (3.4)$$

$$[\partial^2 f / \partial \langle n \rangle^2]_{\lambda_c \langle n \rangle_c, \beta_c} = 0. \quad (3.5)$$

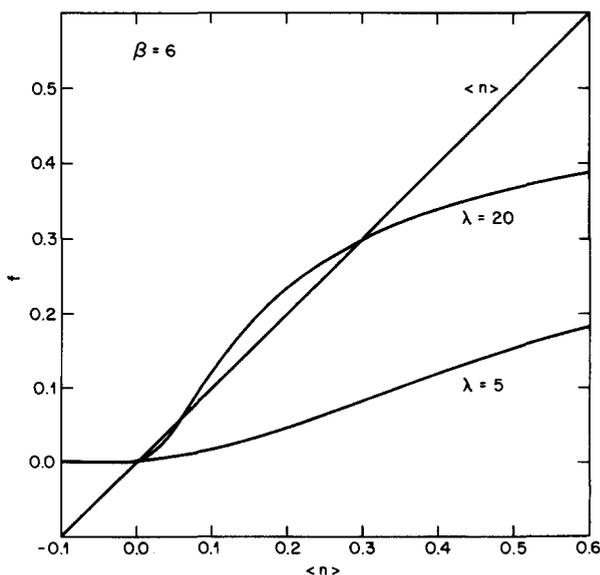
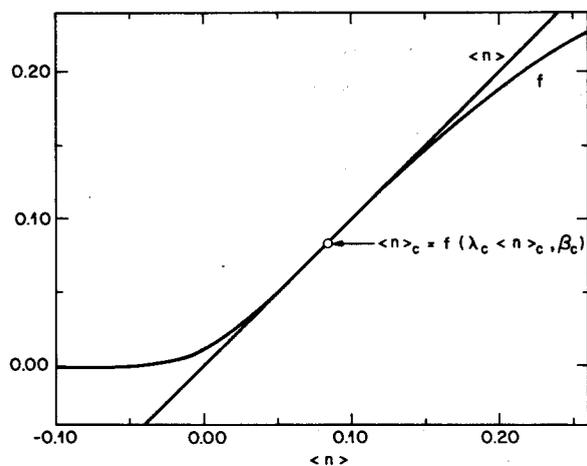


FIG. 1. Plots of both members of Eq. (3.1) for $\beta = 6$.

FIG. 2. Plots of both members of Eq. (3.1) for λ_c, β_c .

By invoking form (3.2) for f , and then using Eq. (3.3) to simplify results, Eqs. (3.4) and (3.5), respectively, can be put into the following forms:

$$\frac{\beta_c}{(1 + \lambda_c \langle n \rangle_c)^2} = \frac{1}{\langle n \rangle_c (1 - \langle n \rangle_c)}, \quad (3.6)$$

$$\frac{\lambda_c}{1 + \lambda_c \langle n \rangle_c} = \frac{1 - 2\langle n \rangle_c}{2\langle n \rangle_c (1 - \langle n \rangle_c)}. \quad (3.7)$$

These lead in turn to the relation

$$\frac{\beta_c}{1 + \lambda_c \langle n \rangle_c} = \frac{2}{1 - 2\langle n \rangle_c}, \quad (3.8)$$

which then permits Eq. (3.3) to be recast as a transcendental equation in $\langle n \rangle_c$ alone:

$$\langle n \rangle_c = \{1 + \exp[2/(1 - 2\langle n \rangle_c)]\}^{-1}. \quad (3.9)$$

The last equation (3.9) possesses only a single real solution in the physically relevant interval $[0, 1]$. Numerical analysis yields the result

$$\langle n \rangle_c = 0.083\ 221\ 720\ 20. \quad (3.10)$$

Using this value, Eq. (3.7) leads to λ_c :

$$\lambda_c = 1/\langle n \rangle_c - 2 = 10.016\ 093\ 85. \quad (3.11)$$

Finally, β_c can be evaluated from Eq. (3.8):

$$\beta_c = 4(1 - \langle n \rangle_c)/(1 - 2\langle n \rangle_c) = 4.399\ 357\ 281. \quad (3.12)$$

Figure 2 exhibits plots vs $\langle n \rangle$ of the two members of Eq. (3.1), where λ and β have been set equal to their critical values λ_c and β_c . The common point of intersection, tangency, and inflection has been explicitly indicated.

Varying λ in the logarithmic model away from the value λ_c in Eq. (3.11) causes the critical point to drift continuously away from $H = 0$. In particular $\lambda > \lambda_c$ moves the critical point to $H > 0$, $\lambda < \lambda_c$ moves it to $H < 0$.

IV. CRITICAL EXPONENTS

Now that the critical point for the logarithmic model has been located, it is straightforward to extract critical exponents for singularities in thermodynamic properties. The required analysis rests upon behavior of solutions to Eq. (2.10) (with $\lambda = \lambda_c$) in the critical region.

Set

$$F(\langle n \rangle, \beta, H) = \left\{ 1 + \exp \left[\frac{\beta}{1 + \lambda_c \langle n \rangle} + H \right] \right\}^{-1}. \quad (4.1)$$

This function can be developed in a locally convergent multiple Taylor series around the critical point

$$F(\langle n \rangle, \beta, H) = \sum_{i,j,k=0}^{\infty} (i!j!k!)^{-1} P(i,j,k) (\Delta n)^i (\Delta \beta)^j H^k, \quad (4.2)$$

where

$$P(i,j,k) = \left[\frac{\partial^i \partial^j \partial^k F}{\partial n^i \partial \beta^j \partial H^k} \right]_{(\langle n \rangle_c, \beta_c, 0)} \quad (4.3)$$

and

$$\begin{aligned} \Delta n &= \langle n \rangle - \langle n \rangle_c, \\ \Delta \beta &= \beta - \beta_c. \end{aligned} \quad (4.4)$$

The partial derivatives of low order are easy to calculate from Eq. (4.1); some of these are

$$\begin{aligned} P(0,0,0) &= \langle n \rangle_c, \\ P(1,0,0) &= 1, \\ P(0,1,0) &= \langle n \rangle_c (\langle n \rangle_c - 1) / (1 + \lambda_c \langle n \rangle_c), \\ P(0,0,1) &= \langle n \rangle_c (\langle n \rangle_c - 1) \beta_c. \end{aligned} \quad (4.5)$$

Precise numerical values for these quantities have been listed in Table I, along with those for a few higher-order derivatives.

Only the leading-order terms from expansion (4.2) are required to infer critical exponents. When just those represented in Table I are retained, the basic relation (2.10) reduces to the following:

$$\begin{aligned} 0 &= \frac{1}{6} P(300) (\Delta n)^3 + P(010) \Delta \beta + P(001) H \\ &+ P(110) \Delta n \Delta \beta + P(101) \Delta n H. \end{aligned} \quad (4.6)$$

First examine the case where $H = 0$, for which this last equation (4.6) yields

$$\Delta \beta = \frac{-P(300) (\Delta n)^3}{6[P(010) + P(110) \Delta n]}. \quad (4.7)$$

In view of the fact that Δn goes continuously to zero as $\Delta \beta$ goes to zero, the leading-order behavior can be obtained by neglecting the term with Δn in the denominator. Thus,

$$\Delta n \cong - [6P(010)/P(300)]^{1/3} (\Delta \beta)^{1/3}, \quad (4.8)$$

valid for both signs of $\Delta \beta$.

Near to the critical point, still at $H = 0$, the mean energy

TABLE I. Values for partial derivatives of F [Eq. (4.2)] at the critical point.

i, j, k	$P(i, j, k)$
0, 0, 0	0.083 221 720 20
1, 0, 0	1.0
0, 1, 0	-0.041 610 860 09
0, 0, 1	-0.335 652 771 3
2, 0, 0	0.0
1, 1, 0	-0.227 305 930 4
1, 0, 1	-3.667 113 120
3, 0, 0	-85.895 022 27

per particle can be written

$$\begin{aligned} \langle E \rangle / N &\cong \frac{\ln(1 + \lambda_c \langle n \rangle_c)}{\lambda_c} + \frac{\Delta n}{1 + \lambda_c \langle n \rangle_c} \\ &= 0.060\,528\,342 - 0.077\,833\,938(\Delta\beta)^{1/3}. \end{aligned} \quad (4.9)$$

The heat capacity C follows immediately:

$$\begin{aligned} C / Nk_B &= -\beta^2 \partial \langle E \rangle / N / \partial \beta \\ &= 0.502\,141\,6166 |\Delta\beta|^{-2/3} + \dots \end{aligned} \quad (4.10)$$

This result demonstrates that the logarithmic model possesses a symmetrical infinite heat capacity anomaly at its critical point, with critical exponent equal to 2/3.

Next we turn to isothermal field dependence of $\langle n \rangle$ at $\Delta\beta = 0$. The analog to the previous equation (4.7) is

$$H = \frac{-P(300)(\Delta n)^3}{6[P(001) + P(101)\Delta n]}, \quad (4.11)$$

and again the Δn term in the denominator can be disregarded. Consequently,

$$\begin{aligned} \Delta n &\cong -[6P(001)/P(300)]^{1/3} H^{1/3} \\ &= -0.286\,214\,1914 H^{1/3}, \end{aligned} \quad (4.12)$$

valid for both signs of the external field. To the extent that we can follow conventional usage,^{4,5} this indicates that the critical isotherm exponent is 3.

In order to extract the critical behavior of the isothermal susceptibility quantity $(\partial \langle n \rangle / \partial H)_\beta$ it suffices to retain just the first three terms in the right member of Eq. (4.6). Therefore,

$$H \cong -\{6[P(010)\Delta\beta + P(001)H]\}^{1/3}, \quad (4.13)$$

which upon differentiation leads to

$$\begin{aligned} (\partial n / \partial H)_\beta &\cong -\frac{1}{3} P(001) [6/P(300)]^{1/3} \\ &\quad \times [P(010)\Delta\beta + P(001)H]^{-2/3}. \end{aligned} \quad (4.14)$$

When $H = 0$ this leads to the expression

$$\begin{aligned} (\partial n / \partial H)_\beta &\cong -\frac{1}{3} P(001) [6/P(300)P^2(010)]^{1/3} |\Delta\beta|^{-2/3} \\ &= 0.383\,729\,8689 |\Delta\beta|^{-2/3}, \end{aligned} \quad (4.15)$$

which reveals that the critical susceptibility exponent is 2/3.

Finally we turn to consideration of the way by which the Δn discontinuity that is present for $H < 0$ vanishes at the critical point. This attribute provides an analog for magnetization or density coexistence-curve shapes familiar in conventional critical phenomena. Similarly we expect to find a characteristic ‘‘coexistence curve’’ exponent for the Δn discontinuity.

We must now consider explicitly all five terms in the right member of Eq. (4.6), each one of which Table I shows to have a negative coefficient. This combination of terms amounts to a cubic polynomial in Δn , and the curve in the β, H plane along which the Δn discontinuity exists must be a locus along which that polynomial has three distinct real roots. Notice that the second and third terms on the right in Eq. (4.6) are the only ones linear in the variables which vanish at the critical point, the remaining three terms being higher order in those variables. Furthermore these linear

terms do not involve Δn , but only $\Delta\beta$ and H . In order for the cubic Δn polynomial to have three real roots upon approach to the critical point, it is necessary for those linear terms to cancel in leading order:

$$P(010)\Delta\beta + P(001)H = 0. \quad (4.16)$$

This relation determines the limiting direction of the discontinuity locus in the β, H plane:

$$\Delta\beta \cong -[P(001)/P(010)]H = -8.066\,470\,402\,H. \quad (4.17)$$

Consequently, $\Delta\beta$ can be eliminated from Eq. (4.6) to yield $\frac{1}{3}P(300)(\Delta n)^2 + \{P(101) - [P(110)P(001)/P(010)]\}H = 0$,

$$(4.18)$$

which leads to the following result for the simultaneous pairs of Δn values at the discontinuity locus:

$$\Delta n \cong \pm 0.357\,881\,1157 |H|^{1/2}, \quad (H < 0). \quad (4.19)$$

Therefore, the ‘‘coexistence exponent’’ is 1/2.

Accurate numerical studies of the logarithmic model (in the infinite-system-size limit) have been carried out in the neighborhood of its critical point. These show not only that the critical-region limiting laws (4.10), (4.12), (4.15), (4.17), and (4.19) are valid, but that they remain reasonable approximations a nonnegligible distance away from the critical point.

V. FINITE SYSTEM COOPERATIVE BEHAVIOR

The mathematical singularities in thermodynamic functions that signify phase change refer strictly only to the infinite system limit. Nevertheless, it is illuminating to observe how these singularities develop as the number of degrees of freedom N grows to infinity. We now study specifically the growth with N of the infinite heat capacity singularity, Eq. (4.10), for the logarithmic model.

An expression for finite- N heat capacity emerges from double β differentiation of the logarithm of Z in Eq. (2.6). It

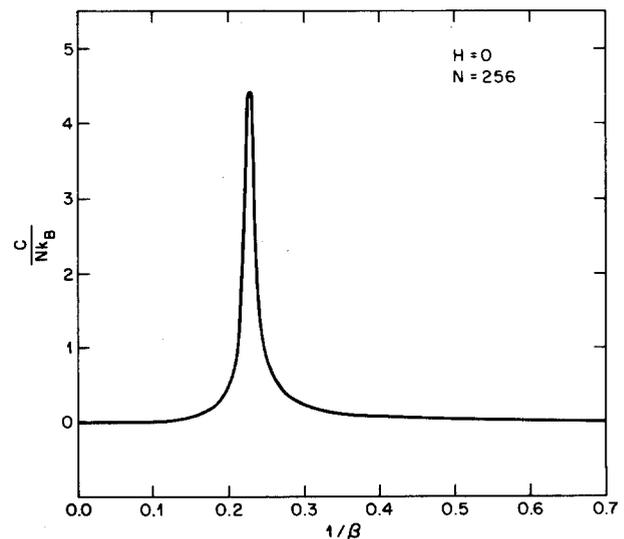


FIG. 3. Heat capacity per degree of freedom for the logarithmic model when $H = 0$, $N = 256$.

TABLE II. Heat capacity maxima for the logarithmic model at $H = 0$, with several system sizes.

N	β (max)	C (max)/ Nk_B
2^0	10.02	0.4392
2^1	7.243	0.4964
2^2	5.728	0.6012
2^3	4.956	0.7824
2^4	4.6053	1.0746
2^5	4.4649	1.5184
2^6	4.4155	2.1679
2^7	4.4006	3.1001
2^8	4.3973	4.4265

has the usual fluctuation form

$$C(\beta, H)/Nk_B = N\beta^2 \{ \langle [(M/N)[\phi(M/N) + H]]^2 \rangle - \langle (M/N)[\phi(M/N) + H] \rangle^2 \}, \quad (5.1)$$

where as before the angular brackets denote a canonical average:

$$\langle B \rangle = Z^{-1} \sum_{M=0}^N B(M) [N!/M!(N-M)!] \times \exp\{-\beta M [\phi(M/N) + H]\}. \quad (5.2)$$

Since only the single summation index M is involved it is relatively easy to subject Eq. (5.1) to direct numerical evaluation for modest values of N .

Figure 3 shows a plot of C/Nk_B for the logarithmic model at $H = 0$, and $N = 256$. A distinct but finite peak appears, centered close to the critical temperature for the infinite system. Similar calculations have been carried out for other smaller values of N , showing lower and broader peaks. Table II provides peak heights and positions for system sizes ranging over several powers of 2.

When β and H are such that the system is off the coexistence locus, fluctuations in M about the mean $\langle M \rangle \equiv \langle n \rangle N$ are normal, i.e., Gaussian with width proportional asymptotically to $N^{1/2}$. On the coexistence locus but away from the critical point, the M distribution is bimodal with separation asymptotically proportional to N . The two portions of the bimodal distribution separately are essentially Gaussian with normal ($N^{1/2}$) widths.

The "critical point" for large finite N can be viewed as the point of confluence of the two Gaussian components of the bimodal distribution. Consequently the M distribution at the heat capacity maximum should have substantially the following non-Gaussian form:

$$\exp\{-AN[(M/N) - \langle n \rangle_c]^4\}, \quad (5.3)$$

where A is a positive constant independent of N . This observation allows direct deduction of the asymptotic N dependence of the heat capacity maximum. It implies that the fluctuation expression (5.1) reduces at the critical point to

$$\frac{N \int x^2 \exp\{-ANx^4\} dx}{\int \exp\{-ANx^4\} dx} \propto N^{1/2}, \quad (5.4)$$

where some irrelevant positive coefficients have been suppressed.

The last equation provides the desired growth law. It implies that if N were to increase from 2^m to 2^{m+2} (with m a sufficiently large positive integer), the height of the heat capacity maximum would double. Entries in Table II show that this behavior is nearly obeyed even when m is as small as 4.

VI. DISCUSSION

Figure 4 indicates the qualitative shape of the surface relating simultaneous values of β , H , and $\langle n \rangle$ at thermal equilibrium. It is instructive to compare this surface to one appropriate, say, to liquid-vapor transitions in the case of fluids, where the three relevant variables are temperature, density, and pressure. The ruled surface of coexistence in Fig. 4 (the area enclosed by the coexistence curve) is oriented in such a way that external field H in the logarithmic model plays the role of temperature in the fluid equation of state. Furthermore, the inverse temperature variable β in Fig. 4 plays a role analogous to that of pressure in the equation of state context. In view of these features, the surface depicted in Fig. 4 resembles nothing more than a rotated version of a classical, or van der Waals-like, equation of state.

Critical singularities and their exponents for classical fluids are well known, and are the same as those encountered in mean-field models of magnetism and of order-disorder phenomena. In particular, the critical-point heat capacity remains bounded in these classical or mean-field equations of state. That the present logarithmic model displays an infinite heat capacity anomaly (in spite of its mean-field character) is not surprising however, because the direction of approach to the critical point, and consequently the direction of the partial derivatives involved in heat capacity, is orthogonal to that of the conventional case. The specific divergence

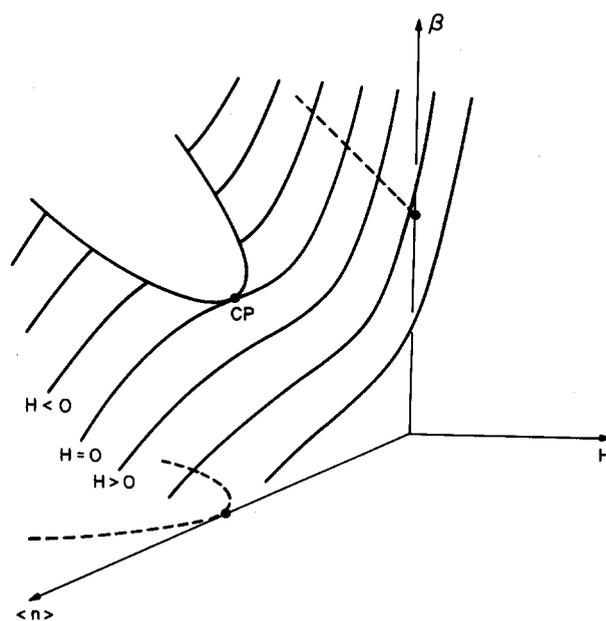


FIG. 4. Schematic representation of the logarithmic-model surface relating simultaneous values of β , H , and $\langle n \rangle$ at thermal equilibrium.

exponent $-2/3$, Eq. (4.10), can be seen to be related to the fact that the critical isotherm degree in the classical equation of state is 3. Similar comments apply to other singularities examined in Sec. IV.

The concept of spatial dimension is irrelevant to the class of models introduced in this paper because all two-level degrees of freedom are uniformly coupled to one another. A nontrivial variant might be considered in which the two-level degrees of freedom were embedded on a regular lattice in D -dimensional Euclidean space, with each two-level system interacting only with those N_0 others within some fixed finite interaction radius R . The energy levels would have the following form:

$$E(n_1 \dots n_N) = \sum_{j=1}^N n_j \left[\phi \left(N_0^{-1} \sum_{l(j)} n_l \right) + H \right] \quad (6.1)$$

which can be compared to the earlier Eq. (2.1). Here the inner sum covers only the interaction zone of neighbors surrounding j . With suitable choice of the coupling function ϕ we can still expect to have phase transitions, even with a critical point at $H = 0$.

The previous case corresponds to allowing the interaction radius R to become indefinitely large. It seems reasonable to expect as well that the same limit obtains if the embedding space dimension D (and along with it the number of neighbors N_0 within a fixed interaction radius) likewise becomes indefinitely large. However, when both D and R are small (for example $D = 3$, and just nearest-neighbor shell coupling) then presumably the critical exponents will differ

from those derived in Sec. IV. It would be interesting to see how critical singularities depend separately on D and on R .

Since the state variables n_i have only the two values 0 and 1, we can write

$$\phi \left(N_0^{-1} \sum_{l(j)} n_l \right) \equiv A_0 + A_2 \sum n_l n_m + \dots + A_{N_0} n_1 \dots n_z, \quad (6.2)$$

where the successive sums are symmetric combinations of all possible subsets of the neighbors of j . The A_k 's are uniquely defined numerical constants that emerge from the requirement that Eq. (6.2) be an identity for any combination of zeros and ones among the neighbors of j . Upon inserting Eq. (6.2) into Eq. (6.1) we see that the Hamiltonian involves one-body, two-body, ..., $(N_0 + 1)$ -body interactions. Consequently, the limits of diverging D or R generate infinitely many-body interactions. Yet in spite of this seeming complication, we have seen that the critical behavior of the limiting class of models is indeed quite simple.

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