# Axiomatic basis for spaces with noninteger dimension

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Five structural axioms are proposed which generate a space $\mathcal{S}_D$ with "dimension" $D$ that is not restricted to the positive integers. Four of the axioms are topological; the fifth specifies an integration measure. When $D$ is a positive integer, $\mathcal{S}_D$ behaves like a conventional Euclidean vector space, but nonvector character otherwise occurs. These $\mathcal{S}_D$ conform to informal usage of continuously variable $D$ in several recent physical contexts, but surprisingly the number of mutually perpendicular lines in $\mathcal{S}_D$ can exceed $D$. Integration rules for some classes of functions on $\mathcal{S}_D$ are derived, and a generalized Laplacian operator is introduced. Rudiments are outlined for extension of Schrödinger wave mechanics and classical statistical mechanics to noninteger $D$. Finally, experimental measurement of $D$ for the real world is discussed.

## I. INTRODUCTION

Continuous variation in the number of dimensions $D$ for space emerges as a useful concept in several areas of physics. It was first introduced, apparently, to aid in understanding critical phenomena exhibited by the binary fluid of "Gaussian molecules."[1] More recently, expansions for critical exponents in terms of 4-D have been developed for a wide range of cooperative many-body systems.[2,3] In addition, quantum field theory has been studied as a function of $D$, which then serves as a regularizing parameter.[4–6] Finally, atomic bound states (as described by the Schrödinger equation) have been studied for continuously variable $D$.[7]

In each of the cited examples, extending $D$ from the positive integers to the real line (or complex plane) has been an obvious procedure advertised by the way that $D$ occurs in certain key quantities. Typical such quantities would be the Gaussian integral

$$\int d\mathbf{r} \exp(-\alpha r^2) = (\pi/\alpha)^{D/2}, \tag{1.1}$$

or the radial Laplace operator

$$\frac{d^2}{dr^2} + \frac{(D-1)}{r}\frac{d}{dr}, \tag{1.2}$$

wherein precisely the same form can be adopted for the extended $D$ domain. Of course the extension is not unique, since one can always augment a given interpolation formula with extra terms which vanish at the positive integers. But regardless of which forms for extension of the key quantities are selected, one must be concerned about their logical independence as assumptions, or even about their logical compatibility.

This paper presents a mathematically concrete realization of spaces with noninteger $D$. In fact, the formalism shows that the specific expressions (1.1) and (1.2) as interpolations are indeed compatible. The broader aim is to provide systematic rules for computation in spaces with noninteger $D$. In the interests of future application to physical theory, we indicate how Schrödinger wave mechanics and Gibbsian statistical mechanics transform into the general-$D$ regime.

The concrete realization offered here may encourage new results in the areas of physics which originally motivated it. The theory of critical phenomena seems to be a good candidate. In particular, convergence properties of critical-exponent expansions in 4-D are uncertain at present. But now that statistical mechanics takes more tangible form for noninteger $D$, it becomes clearer how one might formulate and attempt to prove perturbation convergence theorems for expansions in 4-D, at least for some domain of positive values for this parameter.

Even leaving aside trivial modifications [such as replacement of $D$ by $D+0.1\sin(\pi D)$ in the interpolation formulas], the formalism offered here for noninteger $D$ may not be unique. Nevertheless, it appears to combine simplicity and utility in a way not easily challenged by alternative approaches. Furthermore, the present formalism is attractive on account of the rich opportunities it displays for pure mathematics; in particular the geometry of sphere packings for noninteger $D$ becomes a valid area for inquiry.[8]

## II. TOPOLOGICAL ASSUMPTIONS

We let $\mathcal{S}_D$ denote the space of interest. It contains points $\mathbf{x}, \mathbf{y}, \cdots$, and has topological structure specified by the following axioms:

A1. *$\mathcal{S}_D$ is a metric space.*

A2. *$\mathcal{S}_D$ is dense in itself.*

A3. *$\mathcal{S}_D$ is metrically unbounded.*

The distance between points $\mathbf{x}$ and $\mathbf{y}$ implied by A1. will be written as $r(\mathbf{x},\mathbf{y})$. It must satisfy the conventional criteria required of metrics[9]:

$$\begin{aligned}
&\text{(a) } r(\mathbf{x},\mathbf{y}) \geq 0,\\
&\text{(b) } r(\mathbf{x},\mathbf{y}) = r(\mathbf{y},\mathbf{x}),\\
&\text{(c) } r(\mathbf{x},\mathbf{x}) = 0,\\
&\text{(d) if } r(\mathbf{x},\mathbf{y}) = 0, \text{ then } \mathbf{x}=\mathbf{y},\\
&\text{(e) } r(\mathbf{x},\mathbf{y}) + r(\mathbf{x},\mathbf{z}) \geq r(\mathbf{y},\mathbf{z}) \text{ (triangle inequality).}
\end{aligned} \tag{2.1}$$

The existence of a metric for $\mathcal{S}_D$ permits neighborhoods of given positive radius to be constructed about each point. That $\mathcal{S}_D$ is dense, Axiom A2., simply means that every such neighborhood about an arbitrary point $\mathbf{x} \in \mathcal{S}_D$ contains at least one other point $\mathbf{y}$. Axioms A1. and A2. together require that $\mathcal{S}_D$ contain an infinite number of points.

  

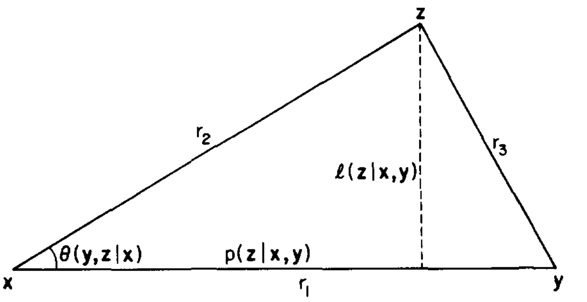


FIG. 1. Definition of geometric quantities for triangles.

Axiom A3. means that for every $\mathbf{x}\in \mathcal{S}_D$, and any $R>0$, there exists a point $\mathbf{y}$ such that

$$r(\mathbf{x},\mathbf{y})>R, \tag{2.2}$$

i.e., the space is infinite in extent.

Real or complex-valued functions $f(\mathbf{x})$ can be defined on $\mathcal{S}_D$. If we have $(i=1,2,3,\cdots)$

$$\lim_{i\to\infty} f(\mathbf{x}_i)=f(\mathbf{x}) \tag{2.3}$$

for all sequences $\{\mathbf{x}_i\}$ with the property

$$\lim_{i\to\infty} r(\mathbf{x}_i,\mathbf{x})=0, \tag{2.4}$$

then $f$ is continuous at $\mathbf{x}$. Similar statements apply to continuity of functions of several variables.

Since any three points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ define three distances obeying the triangle inequality, it will be convenient to adopt some familiar geometrical results for triangles (see Fig. 1). In particular, the angle $0\leqslant\theta(\mathbf{y},\mathbf{z}|\mathbf{x})\leqslant\pi$ subtended by $\mathbf{y}$ and $\mathbf{z}$ at $\mathbf{x}$ can be obtained from the "cosine law,"

$$\cos\theta(\mathbf{y},\mathbf{z}|\mathbf{x})=\frac{r_1^2+r_2^2-r_3^2}{2r_1r_2}, \tag{2.5}$$

where

$$r_1\equiv r(\mathbf{x},\mathbf{y}),\quad r_2\equiv r(\mathbf{x},\mathbf{z}),\quad r_3\equiv r(\mathbf{y},\mathbf{z}). \tag{2.6}$$

This definition leads immediately to expressions for the "projection of $\mathbf{z}$ along $(\mathbf{x},\mathbf{y})$," written $p(\mathbf{z}|\mathbf{x},\mathbf{y})$, as well as its orthogonal complement $l(\mathbf{z}|\mathbf{x},\mathbf{y})$:

$$p(\mathbf{z}|\mathbf{x},\mathbf{y})=r(\mathbf{x},\mathbf{z})\cos\theta(\mathbf{y},\mathbf{z}|\mathbf{x})=\frac{r_1^2+r_2^2-r_3^2}{2r_1}, \tag{2.7}$$

$$l(\mathbf{z}|\mathbf{x},\mathbf{y})=\frac{1}{2r_1}\left[2(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2)-r_1^4-r_2^4-r_3^4\right]^{1/2}, \tag{2.8}$$

$$r_2^2=p^2+l^2. \tag{2.9}$$

In ordinary Euclidean spaces, vector addition is permitted,

$$\mathbf{u}=a\mathbf{x}+b\mathbf{y}, \tag{2.10}$$

and the result is again an element of the space. We must specifically reject (2.10) for noninteger $D$, since any vector space must have a finite integer, or infinite, number of basis vectors,[10] and that number inevitably becomes the space dimension. Hence $\mathcal{S}_D$ normally will not be a vector space.

Again for Euclidean vector spaces, a triangle formed from three points as shown in Fig. 1 has sides (and altitude $l$) which are themselves embedded in the space. This obviously follows from the fact that any point on a line can be expressed as a linear combination of the endpoints, in the form of Eq. (2.10). But for our noninteger-$D$ spaces, the available axioms A1., A2., and A3. are insufficient to ensure that *any* points of $\mathcal{S}_D$ lie precisely between two triangle vertices, let alone an entire side.

To rectify matters, it will be necessary to include a fourth axiom:

A4. *For any two points* $\mathbf{y},\mathbf{z}\in\mathcal{S}_D$, *and any* $\epsilon>0$, *there exists an* $\mathbf{x}\in\mathcal{S}_D$ *such that*:

(a) $r(\mathbf{x},\mathbf{y})+r(\mathbf{x},\mathbf{z})=r(\mathbf{y},\mathbf{z})$;

(b) $|r(\mathbf{x},\mathbf{y})-r(\mathbf{x},\mathbf{z})|<\epsilon r(\mathbf{y},\mathbf{z})$.

Part (a) permits equality always to be achieved in the triangle inequality; an equivalent phrasing would be that $\theta(\mathbf{y},\mathbf{z}|\mathbf{x})=\pi$. Either way, it places an $\mathbf{x}$ directly between $\mathbf{y}$ and $\mathbf{z}$. Part (b) permits $\mathbf{x}$ to be near the midpoint. The full implication of A4. is that any two points in $\mathcal{S}_D$ are connected by a continuous line embedded in that space.

## III. INTEGRATION MEASURE

The topological structure imposed on $\mathcal{S}_D$ must now be supplemented with a statement of volume element size, so that a linear integration operation becomes possible. Considering the fact that, thus far, only points and distances exist for $\mathcal{S}_D$, we are obliged to introduce weights,

$$W_n(\mathbf{x}_1\cdots\mathbf{x}_n|r_1\cdots r_n) \tag{3.1}$$

for a fixed set of points $\mathbf{x}_1\cdots\mathbf{x}_n$, and distances $r_1\cdots r_n$ measured from them. If thin "spherical" shells (with inner and outer radii $r_1$ and $r_1+dr_1$, $r_2$ and $r_2+dr_2,\cdots$) are erected respectively about $\mathbf{x}_1\cdots\mathbf{x}_n$, then $W_n\,dr_1\cdots dr_n$ gives the content of the mutual intersection of those shells. Once having the $W_n$ in hand, it becomes possible to integrate functions $h(r_{01}\cdots r_{0n})$ of the distances $r_{0j}\equiv r(\mathbf{x}_0,\mathbf{x}_j)$ over all $\mathbf{x}_0\in\mathcal{S}_D$ by the simple expedient of using the $r_{0j}$ as separate conventional integration variables,

$$\begin{aligned}\int d\mathbf{x}_0\,h(r_{01}\cdots r_{0n})&=\int_0^\infty dr_{01}\cdots\int_0^\infty dr_{0n}\,W_n(\mathbf{x}_1\cdots\mathbf{x}_n|r_{01}\cdots r_{0n})\\&\quad\times h(r_{01}\cdots r_{0n}).\end{aligned} \tag{3.2}$$

Repeated application of this general procedure would permit evaluation of multiple integrals, over several $\mathbf{x}_j$'s in a finite point set, of functions of distances in that set.

In principle, explicit formulas could be provided for the $W_n$ as functions of the $\frac{1}{2}n(n+1)$ distances $r_{ij}$ $(0\leqslant i\leqslant j\leqslant n)$. In practice, it is more efficient to define those functions implicitly by demanding that multiply-rooted Gaussian integrals have preassigned values. Consequently, we now state the fifth axiom for $\mathcal{S}_D$:

A5. *For any positive integer n,*

$$\int d\mathbf{x}_0 \exp\left(-\sum_{j=1}^{n} \alpha_j r_{0j}^2\right) = \left(\frac{\pi}{\tau}\right)^{D/2} \exp\left(-\frac{1}{\tau} \sum_{j<k=1}^{n} \alpha_j\alpha_k r_{jk}^2\right), \quad \tau = \sum_{j=1}^{n} \alpha_j. \tag{3.3}$$

This is the only point at which the dimension parameter $D$ enters the present axiomatic formalism. It should be noted that (3.3) coincides with standard integral results when $D$ is a positive integer. When $n=1$, Eq. (3.3) agrees with Eq. (1.1).

Axiom A5. confers overall uniformity on $\mathcal{S}_D$. The result produced by integrating any rooted Gaussian depends only on distances between root points (which can be anywhere in $\mathcal{S}_D$), and not in any way on absolute position in $\mathcal{S}_D$. In this sense there are no distinguished points in $\mathcal{S}_D$.

By combining Eqs. (3.2) and (3.3), along with the variable change

$$t_j = r_{0j}^2, \tag{3.4}$$

one discovers the identities

$$\int_0^\infty dt_1 \cdots \int_0^\infty dt_n \left[\frac{W_n(\cdots | t_1^{1/2} \cdots t_n^{1/2})}{2^n (t_1 \cdots t_n)^{1/2}}\right] \times \exp\left(-\sum_{j=1}^{n} \alpha_j t_j\right) = \left(\frac{\pi}{\tau}\right)^{D/2} \exp\left(-\frac{1}{\tau} \sum_{j<k=1}^{n} \alpha_j\alpha_k r_{jk}^2\right). \tag{3.5}$$

Essentially, this provides the result of an $n$-fold Laplace transform on $W_n$. The weight itself can be computed from the appropriate transform inversion formula[11]

$$W_n(\cdots | t_1^{1/2} \cdots t_n^{1/2}) = \frac{(t_1 t_2 \cdots t_n)^{1/2}}{(\pi i)^n} \int_{c_1 - i\infty}^{c_1 + i\infty} d\alpha_1 \cdots \int_{c_n - i\infty}^{c_n + i\infty} d\alpha_n \left(\frac{\pi}{\tau}\right)^{D/2} \times \exp\left(\sum_{j=1}^{n} t_j \alpha_j - \frac{1}{\tau} \sum_{j<k=1}^{n} r_{jk}^2 \alpha_j \alpha_k\right). \tag{3.6}$$

The simplest of the weights, $W_1$, allows integrals of radially symmetric functions to be computed:

$$\int d\mathbf{x}_0 f[r(\mathbf{x}_0, \mathbf{x}_1)] = \int_0^\infty dr\, W_1(r) f(r). \tag{3.7}$$

The inverse Laplace transform needed to find $W_1$ from Eq. (3.6) is a standard form.[12] The result is found to be

$$W_1(r) = \sigma(D) r^{D-1}, \quad \sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{3.8}$$

When $D$ is a positive integer this agrees precisely with the known spherical volume element for mutlidimensional Euclidean spaces.[13] This is the central fact which justifies the claim that $\mathcal{S}_D$ is a "space of $D$ dimensions."

The volume of the radius-$R$ sphere in $\mathcal{S}_D$ follows immediately from $W_1$,

$$V(R, D) = \int_0^R W_1(r)\, dr = \frac{\pi^{D/2} R^D}{\Gamma(1 + \frac{1}{2}D)}. \tag{3.9}$$

For any $R > 0$ this has the property

$$\lim_{D\to 0} V(R, D) = 1, \tag{3.10}$$

which in turn implies that $W_1$ is a Dirac delta function in the same limit,

$$\lim_{D\to 0} W_1(r) = \delta(r - 0). \tag{3.11}$$

Consequently, integration weight in $\mathcal{S}_D$ collapses to zero extension, in spite of the fact that Axiom A3. still maintains a sparse set of widely separated pairs of points. For a continuous function $f$,

$$\lim_{D\to 0} \int d\mathbf{x} f(r) = f(0). \tag{3.12}$$

Equation (3.6) may be used to derive a consistency property of the weights,

$$\int_0^\infty dr_n\, W_n(\mathbf{x}_1 \cdots \mathbf{x}_n | r_1 \cdots r_n) = W_{n-1}(\mathbf{x}_1 \cdots \mathbf{x}_{n-1} | r_1 \cdots r_{n-1}). \tag{3.13}$$

## IV. DENSITY OF MUTUALLY PERPENDICULAR LINES

Inverting the Laplace transforms, as required by Eq. (3.6) to obtain $W_n$, becomes an increasingly arduous task as $n$ increases. But experience shows that no insuperable difficulties arise—one needs recourse only to a small number of recurrent tabulated inverse-transfrom types.

One finds the following expression for the two-center weight (valid for all real $D$):

$$W_2(\mathbf{x}_1, \mathbf{x}_2 | r_{01}, r_{02}) = 2^{D-3}\sigma(D-1) r_{01} r_{02} r_{12}^{2-D} \Delta^{D-3}(r_{01}, r_{02}, r_{12}), \tag{4.1}$$

where $\Delta$ is the area of the triangle having sides $r_{01}$, $r_{02}$, and $r_{12}$,

$$\Delta(r_{01}, r_{02}, r_{12}) = \tfrac{1}{4}[2(r_{01}^2 r_{02}^2 + r_{01}^2 r_{12}^2 + r_{02}^2 r_{12}^2) - r_{01}^4 - r_{02}^4 - r_{12}^4]^{1/2}. \tag{4.2}$$

If no triangle can be formed, $\Delta$ must be set equal to zero. By setting $D=3$, expression (4.1) reduces to a familiar weight for the nonorthogonal bipolar coordinate system,

$$W_2(D=3) = 2\pi r_{01} r_{02} / r_{12}. \tag{4.3}$$

A right triangle will be formed if $r_{01} = r_{02} = R$, $r_{12} = 2^{1/2} R$, with the right angle at vertex 0. By inserting these values in $W_2$ we obtain a measure for the density of mutually perpendicular lines,

$$W_2(2^{1/2} R | R, R) = 2^{2-D/2} \pi^{(D-1)/2} R^{D-2} / \Gamma\left(\frac{D-1}{2}\right). \tag{4.4}$$

This result is positive for all $D > 1$. It leads to the striking conclusion that the number of mutually perpendicular lines can exceed the dimension of a space, specifically when $2 > D > 1$.

Strictly speaking, we have not proven that triplets of points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ exist with connecting lines at exactly a right angle. The result on a density being positive in the neighborhood of this configuration is a weaker statement. However the spaces $\mathcal{S}_D$ are dense, so the distinction for most purposes is unimportant.

The three-center weight has the following lengthy form (valid for all real $D$):

$$W_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 | r_{01}, r_{02}, r_{03})$$
$$= 2^{7-2D}\sigma(D-2)\, r_{01}r_{02}r_{03}[\Delta(r_{12}, r_{13}, r_{23})]^{3-D}$$
$$\times\{-r_{12}^2r_{13}^2r_{23}^2 + r_{01}^2(r_{12}^2r_{23}^2 + r_{13}^2r_{23}^2 - r_{23}^4)$$
$$+ r_{02}^2(r_{12}^2r_{13}^2 + r_{13}^2r_{23}^2 - r_{13}^4) + r_{03}^2(r_{12}^2r_{13}^2 + r_{12}^2r_{23}^2 - r_{12}^4)$$
$$- r_{01}^4r_{23}^2 - r_{02}^4r_{13}^2 - r_{03}^4r_{12}^2 + r_{01}^2r_{02}^2(r_{13}^2 + r_{23}^2 - r_{12}^2)$$
$$+ r_{01}^2r_{03}^2(r_{12}^2 + r_{23}^2 - r_{13}^2) + r_{02}^2r_{03}^2(r_{12}^2 + r_{13}^2 - r_{23}^2)\}^{(D-4)/2}. \tag{4.5}$$

The density of mutually perpendicular lines in $\mathcal{S}_D$ can be extracted from this formula upon setting $r_{01} = r_{02} = r_{03} = R$, and $r_{12} = r_{13} = r_{23} = 2^{1/2}R$. This yields

$$W_3(2^{1/2}R \cdots | R \cdots) = 2\pi^{(D-2)/2}R^{D-3}/3^{(D-3)/2}\,\Gamma\left(\frac{D-2}{2}\right), \tag{4.6}$$

indicating a positive density for all $D > 2$. Analogous to the preceding case, we have found that the number of mutually perpendicular lines can exceed the dimension. The orthodox position that the maximal number of mutually perpendicular lines gives $D$ is not valid in the present context.

Careful study of the $W_n$, using Eq. (3.6), shows that they always consist of nonnegative factors divided by an uncompensated term $\Gamma[\frac{1}{2}(D-n+1)]$. When sets of distances serving as arguments for $W_n$ are chosen so that this weight does not vanish identically, then the incorporated term $1/\Gamma[\frac{1}{2}(D-n+1)]$ will cause sign alternation as a function of $D$ when $D < n-1$. In particular, one will have

$$W_n < 0 \quad (n-4j-3 < D < n-4j-1),$$
$$= 0 \quad (D = n-2j-1), \tag{4.7}$$

where $j = 0, 1, 2, 3, \cdots$. For all other values of $D$, $W_n$ will be positive. The occurrence of negative integration weights for noninteger $D$ indicates a complicated and unanticipated "geometric" structure for the $\mathcal{S}_D$. In fact with finite noninteger $D$ an arbitrary number of mutually perpendicular lines can be erected, though the corresponding weights $W_n(2^{1/2}R \cdots | R \cdots)$ will have indefinite signs. The possibility of continuously variable $D$ evidently has been bought at the expense of negative integration weights, which have no precedent in ordinary geometry.

If $M$ lines emanate from point $\mathbf{x}_0$ in $\mathcal{S}_D$, projection of any $\mathbf{x}$ along each of these lines can be computed by means of Eq. (2.7); they might be denoted by

$$p_1(\mathbf{x}),\ p_2(\mathbf{x}), \ldots, p_M(\mathbf{x}). \tag{4.8}$$

Provided that $D$ is not an integer, the $M$ lines can be chosen to be arbitrarily close to perpendicular to one another, regardless of how large $M$ might be. These lines can then be regarded as a set of orthogonal axes along which the "pseudocoordinates" $p_1 \cdots p_M$ are measured. There are several fundamental questions about these pseudocoordinates that deserve eventually to be investigated, such as:

(a) If $M \geq D$, and $\mathbf{x} \neq \mathbf{y}$, are the sets of pseudocoordinates for $\mathbf{x}$ and for $\mathbf{y}$ distinct?

(b) Are straight lines in $\mathcal{S}_D$ always representable as linear parametric expressions in terms of pseudocoordinates?

(c) Under what circumstances can pseudocoordinates serve as integration variables?

(d) What is the content ("volume") of the region defined by $0 < p_j < L$, if $M \geq D$?

(e) Can pseudocoordinates be used to carry the concept of convex regions into the noninteger-$D$ regime?

(f) How do pseudocoordinates transform under the translation and rotation groups in $\mathcal{S}_D$?

(g) How can the Pythagorean formula, which for $M = D =$ integer has the form

$$[r(\mathbf{x}, \mathbf{y})]^2 = \sum_{j=1}^{M} [p_j(\mathbf{x}) - p_j(\mathbf{y})]^2, \tag{4.9}$$

be generalized to arbitrary $M$ and $D$? In particular, do $D \leq M$ and $D \geq M$ require corresponding inequalities between the members of Eq. (4.9)?

We shall not consider these open problems any further in the remainder of this exposition.

## V. CONVOLUTION THEOREM

In Euclidean spaces ($D =$ integer), integrals of the type

$$T\{f, h\} = \int d\mathbf{r}_3\, f(r_{13})\, h(r_{23}) \tag{5.1}$$

can be reduced to simpler quadratures by introducing Fourier transforms for the functions $f$ and $h$. The general reduction scheme is usually referred to as the "convolution theorem[14]", whose extension to noninteger $D$ we now identify.

To prepare the way for introduction of Fourier transforms in $\mathcal{S}_D$, it will first be necessary to have an integration weight in terms of quantities $p$ and $l$ [Eqs. (2.7)–(2.9)]. This can be produced from the general doubly-rooted Gaussian integral, which we now write in the following manner:

$$\int d\mathbf{x}_0 \exp(-\alpha_1 r_{01}^2 - \alpha_2 r_{02}^2)$$
$$= \int_{-\infty}^{+\infty} dp \int_0^{\infty} dl\, W(p, l) \exp[-\alpha_1 p^2 - \alpha_2(r_{12} - p)^2$$
$$- (\alpha_1 + \alpha_2) l^2]. \tag{5.2}$$

Here the fixed points 1 and 2 are separated by $r_{12}$, $p \equiv p(\mathbf{x}_0 | \mathbf{x}_1, \mathbf{x}_2)$ is the projection of $(\mathbf{x}_1, \mathbf{x}_0)$ on $(\mathbf{x}_1, \mathbf{x}_2)$, and $l$ is its orthogonal complement.

Since $\mathcal{S}_D$ is uniform, $W$ cannot depend on position $p$ measured along the arbitrary axis passing through $\mathbf{x}_1$ and $\mathbf{x}_2$. This fact permits the $p$ integral in Eq. (5.2) to be carried out explicitly. Furthermore, A5. specifies the value to be assigned to (5.2), so we have

$$\left(\frac{\pi}{\alpha_1+\alpha_2}\right)^{(D-1)/2} = \int_0^\infty dl\, W(p,l)\exp[-(\alpha_1+\alpha_2)l^2]. \tag{5.3}$$

This is equivalent to a Laplace transform, and the inversion operation leads to the result

$$W(p,l)=\sigma(D-1)\,l^{D-2}. \tag{5.4}$$

Comparing this result with Eq. (3.8), we see that $W(p,l)$ is equivalent to a radial weight (with $r=l$) for $\mathcal{S}_{D-1}$. This confirms the expectation that a constant-$p$ subspace in $\mathcal{S}_D$ has dimension $D-1$, and we note in passing that this subspace can be proved to have all other integration properties of the type embodied in A5.

Now we are in a position to evaluate the "Fourier transform" of a Gaussian function in $\mathcal{S}_D$,

$$G(k)=\int d\mathbf{x}\exp[-\alpha r^2(\mathbf{x})+ikp(\mathbf{x})]. \tag{5.5}$$

Here $p(x)$ is the projection along a preselected axis through the origin. Using Eq. (5.4) we have

$$\begin{aligned}G(k)&=\int_{-\infty}^{+\infty} dp\exp(-\alpha p^2+ikp)\int_0^\infty dl\,\sigma(D-1)\,l^{D-2}\exp(-\alpha l^2)\\&=(\pi/\alpha)^{D/2}\exp(-k^2/4\alpha).\end{aligned} \tag{5.6}$$

Identifying parameter $k$ as a distance function $k(\mathbf{x})$ in $\mathcal{S}_D$, with $p(\mathbf{x})$ the corresponding projection, we also derive

$$\begin{aligned}g(r)&=(2\pi)^{-D}\int d\mathbf{x}\,G[k(\mathbf{x})]\exp[-irp(\mathbf{x})]\\&=(4\pi\alpha)^{-D/2}\int d\mathbf{x}\exp\{-[k^2(\mathbf{x})/4\alpha]-irp(\mathbf{x})\}\\&=\exp(-\alpha r^2).\end{aligned} \tag{5.7}$$

This constitutes the inverse to transform Eq. (5.5).

Consider next the class of functions which consist of linear combinations of Gaussians,

$$f(r)=\sum_{j=1}^{m} A_j\exp(-a_j r^2). \tag{5.8}$$

Our generalized Fourier transformation is linear, so that

$$F(k)=\sum_{j=1}^{m} A_j(\pi/a_j)^{D/2}\exp(-k^2/4a_j) \tag{5.9}$$

is the corresponding transform function. At least within this function class, the symbolic Fourier transform pair has the following appearance:

$$F(k)=\int d\mathbf{x}\,f[r(\mathbf{x})]\exp[ikp(\mathbf{x})], \tag{5.10a}$$

$$f(r)=(2\pi)^{-D}\int d\mathbf{x}\,F[k(\mathbf{x})]\exp[-irp(\mathbf{x})]. \tag{5.10b}$$

Define the following two-center integral (fixed points 1 and 2) in $\mathcal{S}_D$:

$$T(r_{12})=\int d\mathbf{x}_0\,f(r_{10})\,h(r_{02}), \tag{5.11}$$

where both $f$ and $h$ belong to the function class denoted by Eq. (5.8). Thus

$$\begin{aligned}T(r_{12})&=\int d\mathbf{x}_0\left[\sum_{j=1}^{m} A_j\exp(-a_j r_{10}^2)\right]\left[\sum_{l=1}^{n} B_l\exp(-b_l r_{02}^2)\right]\\&=\sum_{j=1}^{m}\sum_{l=1}^{n} A_jB_l\int d\mathbf{x}_0\exp(-a_j r_{10}^2-b_l r_{02}^2)\\&=\sum_{j=1}^{m}\sum_{l=1}^{n} A_jB_l\left(\frac{\pi}{a_j+b_l}\right)^{D/2}\exp\left[-\left(\frac{a_jb_l}{a_j+b_l}\right)r_{12}^2\right],\end{aligned} \tag{5.12}$$

by invoking A5. The remaining Gaussian factor may itself be written as an integral,

$$\begin{aligned}&\exp\left[-\left(\frac{a_jb_l}{a_j+b_l}\right)r_{12}^2\right]\\&=\left(\frac{a_j+b_l}{4\pi a_jb_l}\right)^{D/2}\int d\mathbf{x}\exp\left[-\left(\frac{a_j+b_l}{4a_jb_l}\right)k^2(\mathbf{x})\right.\\&\qquad\left.-ir_{12}p(\mathbf{x})\right].\end{aligned} \tag{5.13}$$

Substituting and rearranging we have

$$\begin{aligned}T(r_{12})&=(2\pi)^{-D}\int d\mathbf{x}\exp[-ir_{12}p(\mathbf{x})]\\&\quad\times\left\{\sum_{j=1}^{m}\sum_{l=1}^{n} A_jB_l\left(\frac{\pi}{a_j}\right)^{D/2}\left(\frac{\pi}{b_l}\right)^{D/2}\right.\\&\quad\left.\times\exp\left[-\frac{1}{4}\left(\frac{1}{a_j}+\frac{1}{b_l}\right)k^2(\mathbf{x})\right]\right\}\\&=(2\pi)^{-D}\int d\mathbf{x}\exp[-ir_{12}p(\mathbf{x})]F[k(\mathbf{x})]H[k(\mathbf{x})].\end{aligned} \tag{5.14}$$

This is the desired convolution theorem. Similarly to the case with integer $D$, it reduces the evaluation of doubly-rooted integrals to an integral of the product of Fourier transforms.

Aside from complex exponentials, integrals of the types (5.10) and (5.14) involve only functions of distance. Consequently they may be simplified. Starting with the prototype integral

$$\begin{aligned}I&=\int d\mathbf{x}\,\phi[r(\mathbf{x})]\exp[ikp(\mathbf{x})]\\&\equiv\int_{-\infty}^{+\infty} dp\int_0^\infty dl\,W(p,l)\,\phi(p^2+l^2)\exp(ikp),\end{aligned} \tag{5.15}$$

we introduce the change of variables

$$l=r\sin\Theta,\quad p=r\cos\Theta \tag{5.16}$$

[recall Eqs. (2.5)–(2.9)]. This allows one to express $I$ as follows

$$\begin{aligned}I&=\sigma(D-1)\int_0^\infty dr\int_0^\pi d\Theta\, r^{D-1}(\sin\Theta)^{D-2}\\&\quad\times\exp(ikr\cos\Theta)\,\phi(r),\end{aligned} \tag{5.17}$$

upon using Eq. (5.4) for $W(p,l)$. By expanding the exponential function, the $\Theta$ integral may be carried out explicitly (after recognizing the Bessel function series),

$$I=(2\pi)^{D/2}\int_0^\infty dr\,(kr)^{(2-D)/2}J_{(D-2)/2}(kr)\,\phi(r). \tag{5.18}$$

By employing result (5.18), we simplify the $D$-dimensional Fourier transform pair (5.10) to

$$F(k)=(2\pi)^{D/2}\int_0^\infty dr\,(kr)^{(2-D)/2}J_{(D-2)/2}(kr)\,f(r), \tag{5.19a}$$

$$f(r) = (2\pi)^{-D/2} \int_0^\infty dk\,(rk)^{(2-D)/2} J_{(D-2)/2}(rk)\,F(k). \qquad (5.19b)$$

In analogous fashion, the convolution theorem (5.14) can be written

$$T(r_{12}) = (2\pi)^{-D/2} \int_0^\infty dk\,(r_{12}k)^{(2-D)/2} J_{(D-2)/2}(r_{12}k)\,F(k)\,H(k), \qquad (5.20)$$

for evaluation of the doubly-rooted integral (5.11).

Equations (5.18)—(5.20) are Hankel transforms,[15] with minor modifications. We see that they arise naturally in spaces with fractional dimension. Although we have derived (5.18)—(5.20) only for the limited class of functions shown in Eq. (5.8), consisting of finite sums of Gaussians, it should be clear that extension is possible to convergent sequences of such functions. The available general theory of Hankel transforms[16] is relevant in this connection.

## VI. LAPLACE OPERATOR

For the moment, we revert to the special case that $D$ is a positive integer, so that $\mathcal{S}_D$ can be treated as a conventional vector space. A form of the linear Laplace operator $\nabla^2$ will be constructed which will serve as a convenient device for extension to noninteger $D$.

Introduce a "local weighting function" $w(r)$ with the following properties:

$$\lim_{r\to\infty} w(r) = 0, \qquad (6.1a)$$

$$\int d\mathbf{r}\, w(r) = 0, \qquad (6.1b)$$

$$\int d\mathbf{r}\, r^2 w(r) \equiv w_2 \neq 0. \qquad (6.1c)$$

Then for any function $f(\mathbf{r})$ defined over the vector space, consider the integral (we assume it converges),

$$L(\mathbf{r}_1, \xi) = \xi^{D+2} \int d\mathbf{r}\, w(\xi|\mathbf{r}-\mathbf{r}_1|)\, f(\mathbf{r}), \quad \xi > 0. \qquad (6.2)$$

When $\xi$ is large, the integrand will differ from zero only in the immediate neighborhood of the point $\mathbf{r}_1$. Presuming that $f$ is at least twice differentiable, it would then suffice to represent this function in $L$ by the leading terms in its multiple Taylor expansion about $\mathbf{r}_1$

$$L(\mathbf{r}_1, \xi) = \xi^{D+2} \int d\mathbf{r}\, w(\xi|\mathbf{r}-\mathbf{r}_1|)[f(\mathbf{r}_1) + (\mathbf{r}-\mathbf{r}_1)\cdot\nabla f(\mathbf{r}_1) + \tfrac{1}{2}(\mathbf{r}-\mathbf{r}_1)(\mathbf{r}-\mathbf{r}_1):\nabla\nabla f(\mathbf{r}_1) + \cdots]. \qquad (6.3)$$

In the limit $\xi \to +\infty$, the remainder beyond terms shown should be negligible, so we drop it.

The leading term in Eq. (6.3) vanishes, due to condition (6.1b). The next term (containing $\nabla f$) also vanishes by symmetry. Therefore, we need only examine the quadratic terms, which may now be written out explicitly,

$$L(\mathbf{r}_1, \xi) = \tfrac{1}{2}\xi^{D+2} \sum_{i,j=1}^{D} \frac{\partial^2 f(\mathbf{r}_1)}{\partial x_i\, \partial x_j} \int d\mathbf{r}\, w(\xi|\mathbf{r}-\mathbf{r}_1|) \times (x_i - x_{i1})(x_j - x_{j1}) + \cdots. \qquad (6.4)$$

Here the separate vector components have been denoted by $x_i$, etc. Only those integrals with $i=j$ in (6.4) survive. Since

$$\int d\mathbf{r}\, w(\xi|\mathbf{r}-\mathbf{r}_1|)(x_i - x_{i1})^2 = w_2/(D\xi^{D+2}), \qquad (6.5)$$

we have

$$L(\mathbf{r}_1, \xi) = \frac{w_2}{2D} \sum_{i=1}^{D} \frac{\partial^2 f(\mathbf{r}_1)}{\partial x_i^2} + \cdots \equiv \frac{w_2}{2D} \nabla^2 f(\mathbf{r}_1) + \cdots. \qquad (6.6)$$

In the limit, one has the following identity:

$$\nabla^2 f(\mathbf{r}_1) = \frac{2D}{w_2} \lim_{\xi\to\infty} \xi^{D+2} \int d\mathbf{r}\, w(\xi|\mathbf{r}-\mathbf{r}_1|)\, f(\mathbf{r}). \qquad (6.7)$$

The right side of the last equation immediately suggests the form in which a linear Laplace operator (which we continue to denote by $\nabla^2$) ought to be defined for noninteger $D$,

$$\nabla^2 f(\mathbf{x}_1) = \frac{2D}{w_2} \lim_{\xi\to\infty} \xi^{D+2} \int d\mathbf{x}_2\, w(\xi r_{12})\, f(\mathbf{x}_2), \qquad (6.8)$$

where Eqs. (6.1) are taken over to $\mathcal{S}_D$ in the obvious way. We now explore some implications of this definition.

One of the simplest cases to which Eq. (6.8) can be applied is that in which $f$ depends only on radial distance $r_{02}$ from some origin $\mathbf{x}_0$. For this case the Laplacian to be evaluated will depend only on distance $r_{01}$,

$$\nabla^2 f(r_{01}) = \frac{2D}{w_2} \lim_{\xi\to\infty} \xi^{D+2} \int d\mathbf{x}_2\, w(\xi r_{12})\, f(r_{02}). \qquad (6.9)$$

On account of the (large) scale factor $\xi$ that occurs in the variable for $w$, attention need only be focused on the region of small $r_{12}$. Referring to Fig. 2, we have

$$r_{02} = r_{01}\left\{1 - 2\cos\Theta\left(\frac{r_{12}}{r_{01}}\right) + \left(\frac{r_{12}}{r_{01}}\right)^2\right\}^{1/2} = r_{01}\left\{1 - \cos\Theta\left(\frac{r_{12}}{r_{01}}\right) + \left(\tfrac{1}{2} - \tfrac{1}{2}\cos^2\Theta\right)\left(\frac{r_{12}}{r_{01}}\right)^2 + O\left[\left(\frac{r_{12}}{r_{01}}\right)^3\right]\right\}. \qquad (6.10)$$

This expansion may be used in conjunction with the Taylor expansion for $f$ to yield the following:

$$\nabla^2 f(r_{01}) = \frac{2D}{w_2} \lim_{\xi\to\infty} \xi^{D+2} \int d\mathbf{x}_2\, w(\xi r_{12})\Big[f(r_{01}) - r_{12}\cos\Theta f'(r_{01}) + r_{12}^2\left(\left(\tfrac{1}{2} - \tfrac{1}{2}\cos^2\Theta\right)\frac{f'(r_{01})}{r_{01}} + \tfrac{1}{2}\cos^2\Theta f''(r_{01})\right) + O(r_{12}^3)\Big]. \qquad (6.11)$$

Equation (6.1b) causes the $f(r_{01})$ term in this last expression to vanish; the following term (proportional to

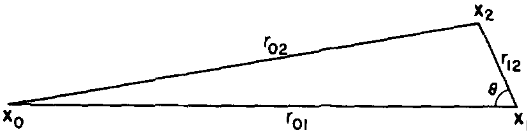


FIG. 2. Triangle used in evaluation of Eq. (6.9).

$r_{12}$) vanishes by symmetry. Furthermore the $O(r_{12}^3)$ remainder will vanish in the indicated limit. Consequently we are left with

$$\nabla^2 f(r_{01}) = \left[ \left\langle \cos^2\Theta \right\rangle f''(r_{01}) + \left(1 - \left\langle \cos^2\Theta \right\rangle\right) \frac{f'(r_{01})}{r_{01}} \right], \tag{6.12}$$

where

$$\langle \cos^2\Theta \rangle = (\xi^{D+2}/w_2) \int d\mathbf{x}_2\, r_{12}^2 \cos^2\Theta\, w(\xi r_{12}). \tag{6.13}$$

It is natural to use $r_{12}$ and $\Theta$ as integration variables for evaluation of this last average. The proper integration weight

$$\sigma(D-1)(r_{12})^{D-1}(\sin\Theta)^{D-2} \tag{6.14}$$

was obtained earlier in connection with Eq. (5.17). Thus we find

$$\langle \cos^2\Theta \rangle = 1 - \langle \sin^2\Theta \rangle = \frac{\int_0^\pi (\sin\Theta)^D d\Theta}{\int_0^\pi (\sin\Theta)^{D-2} d\Theta} = 1/D. \tag{6.15}$$

This converts Eq. (6.12) to the desired Laplacian formula,

$$\nabla^2 f(r) = f''(r) + [(D-1)/r] f'(r), \tag{6.16}$$

where for simplicity the distance subscripts have been suppressed. Note that this confirms the compatibility of expressions (1.1) and (1.2) in the Introduction.

It is only a bit more complicated to compute the Laplacian in $\mathcal{S}_D$ for $g(p, l)$, a function of a projection $p$ and its orthogonal complement. From Eq. (6.8) we have

$$\nabla^2 g[p(\mathbf{x}_1), l(\mathbf{x}_1)] = \frac{2D}{w_2} \lim_{\xi \to \infty} \xi^{D+2} \int d\mathbf{x}_2\, w(\xi r_{12}) \times g[p(\mathbf{x}_2), l(\mathbf{x}_2)]. \tag{6.17}$$

Insert into the integrand the Taylor expansion of $g$ through second order,

$$g[p(\mathbf{x}_2), l(\mathbf{x}_2)] = g[p(\mathbf{x}_1), l(\mathbf{x}_1)] + \frac{\partial g}{\partial p} \Delta p + \frac{\partial g}{\partial l} \Delta l + \frac{1}{2} \frac{\partial^2 g}{\partial p^2} (\Delta p)^2 + \frac{\partial^2 g}{\partial p \partial l} \Delta p \Delta l + \frac{1}{2} \frac{\partial^2 g}{\partial l^2} (\Delta l)^2 + \cdots, \tag{6.18}$$

where

$$\Delta p = p(\mathbf{x}_2) - p(\mathbf{x}_1), \quad \Delta l = l(\mathbf{x}_2) - l(\mathbf{x}_1). \tag{6.19}$$

The first, second, and fifth terms in (6.18) integrate to zero. The remaining three terms may be evaluated by the procedure used to derive the earlier result (6.16). One finally obtains

$$\nabla^2 g(p, l) = \left[ \frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial l^2} + \frac{D-2}{l} \frac{\partial}{\partial l} \right] g(p, l). \tag{6.20}$$

For many applications it may be convenient to transform variables in $g$ from $p$ and $l$ to the polar variables $r$ and $\Theta$,

$$p = r\cos\Theta, \quad l = r\sin\Theta. \tag{6.21}$$

The Laplacian in Eq. (6.20) may be transformed according to the standard procedure[17] to yield

$$\nabla^2 g = \left[ \frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin^{D-2}\theta} \frac{\partial}{\partial \Theta} \sin^{D-2}\Theta \frac{\partial}{\partial \Theta} \right] g. \tag{6.22}$$

## VII. SCHRÖDINGER WAVE MECHANICS

Using suitable reduced units, the quantum-mechanical motion of a particle subject to potential $U$ is described by the time-dependent Schrödinger equation

$$[-\tfrac{1}{2}\nabla^2 + U]\Psi = i \frac{\partial \Psi}{\partial t}. \tag{7.1}$$

The general solution consists of a linear superposition of terms

$$\psi \exp(-iEt), \tag{7.2}$$

where the spatial wavefunctions $\psi$ obey the spatial wave equation

$$[-\tfrac{1}{2}\nabla^2 + U - E]\psi = 0. \tag{7.3}$$

In seeking solutions to Eq. (7.3) relevant to unbounded space, both square-integrable eigenfunctions (bound states) and scattering solutions (asymptotic plane waves) normally are sought.

In view of our generalized Laplace operator, Eq. (6.8), it is now possible to extend study of the Schrödinger wave equation to spaces with noninteger dimension. We examine several simple examples.

Let $U$ be restricted to central form, i.e., it will depend only on radial distance $r$ from some chosen origin in $\mathcal{S}_D$. We then search for solutions to the generalized spatial equation (7.3) which have the form $\psi(r, \theta)$. Here angle $\theta$ is measured relative to *any* axis in $\mathcal{S}_D$ passing through the origin. Appealing to Eq. (6.22), we have

$$\left[ \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin^{D-2}\theta} \frac{\partial}{\partial \theta} \sin^{D-2}\theta \frac{\partial}{\partial \theta} + 2E - 2U(r) \right] \psi(r, \theta) = 0. \tag{7.4}$$

This equation is separable; set

$$\psi(r, \theta) = R(r)\,\Theta(\theta). \tag{7.5}$$

The resulting radial and angular differential equations are the following:

$$\left[ \frac{d^2}{d\theta^2} + (D-2)\cot\theta \frac{d}{d\theta} + \Lambda(\Lambda + D - 2) \right] \Theta(\theta) = 0, \tag{7.6}$$

$$\left[ \frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} + 2E - 2U(r) - \frac{\Lambda(\Lambda + D - 2)}{r^2} \right] R(r) = 0. \tag{7.7}$$

The appropriate solutions to angular Eq. (7.6) are Gegenbauer polynomials in $\cos\theta$[18]:

$$\Theta(\theta) = C_\Lambda^{(D/2-1)}(\cos\theta), \quad \Lambda = 0, 1, 2, 3, \cdots. \tag{7.8}$$

These polynomials satisfy the following orthogonality relation:

$$\int_0^\pi C_\Lambda^{(D/2-1)}(\cos\theta)\, C_{\Lambda'}^{(D/2-1)}(\cos\theta) \sin^{D-2}\theta\, d\theta = h(\Lambda)\, \delta(\Lambda, \Lambda'), \tag{7.9}$$

where

$$h(\Lambda)=\frac{2^{3-D}\pi\Gamma(\Lambda+D-2)}{\Lambda!(\Lambda+\frac{1}{2}D-1)[\Gamma(\frac{1}{2}D-1)]^2}. \tag{7.10}$$

The first few Gegenbauer polynomials are

$$\begin{aligned} C_0^{(D/2-1)}(z)&=1,\\ C_1^{(D/2-1)}(z)&=(D-2)z,\\ C_2^{(D/2-1)}(z)&=(\tfrac{1}{2}D-1)(Dz^2-1). \end{aligned} \tag{7.11}$$

The nature of solutions to the radial equation naturally depends on $U$. The simplest case is that for free-particle motion, $U\equiv 0$. The radial solutions are then found to be expressible in terms of Bessel functions,

$$R(r)=(kr)^{1-D/2}J_{D/2+\Lambda-1}(kr), \quad k=(2E)^{1/2}. \tag{7.12}$$

Free particle motion can just as well be described by a "plane wave" $\psi$. In $\mathcal{S}_D$ the appropriate form is

$$\psi(\mathbf{x})=\exp[ikp(\mathbf{x})], \tag{7.13}$$

where $p(\mathbf{x})$ represents the projection of point $\mathbf{x}$ along the chosen polar axis. This polar axis is the direction of propagation. The "plane wave" may be expanded as follows [recall $p(\mathbf{x})=r(\mathbf{x})\cos\theta$]:

$$\exp[ikp(\mathbf{x})]=\sum_{\Lambda=0}^{\infty}A_\Lambda C_\Lambda^{(D/2-1)}(\cos\theta)(kr)^{1-D/2}J_{D/2+\Lambda-1}(kr),$$

$$A_\Lambda=2^{D/2-1}(\tfrac{1}{2}D+\Lambda-1)\,\Gamma(\tfrac{1}{2}D-1)\,i^\Lambda. \tag{7.14}$$

By choosing

$$U(r)=\tfrac{1}{2}Kr^2 \tag{7.15}$$

we obtain an isotropic harmonic oscillator in $\mathcal{S}_D$. The corresponding discrete spectrum results from the requirement that the radial function $R_n(r)$ vanish at infinity. With this boundary condition the solutions involve generalized Laguerre polynomials,[18]

$$\begin{aligned} R(r)&=\exp(-\tfrac{1}{2}s^2)\,s^\Lambda L_n^{(D/2+\Lambda-1)}(s^2),\\ s&=K^{1/4}r, \quad n=0,1,2,3,\ldots. \end{aligned} \tag{7.16}$$

The corresponding energy eigenvalues are

$$E=K^{1/2}(\tfrac{1}{2}D+\Lambda+2n). \tag{7.17}$$

The lowest-order generalized Laguerre polynomials have the following explicit forms:

$$\begin{aligned} &L_0^{(D/2+\Lambda-1)}(z)=1, \quad L_1^{(D/2+\Lambda-1)}(z)=\tfrac{1}{2}D+\Lambda-z,\\ &L_2^{(D/2+\Lambda-1)}(z)=\tfrac{1}{2}z^2-(\tfrac{1}{2}D+\Lambda+1)z+\tfrac{1}{2}(\tfrac{1}{2}D+\Lambda)(\tfrac{1}{2}D+\Lambda+1). \end{aligned} \tag{7.18}$$

The "Coulomb" problem in $D$ dimensions for present purposes will refer to the inverse-distance potential ($Z\geq 0$)

$$U(r)=-Z/r. \tag{7.19}$$

[An alternative convention might have been adopted, of course, with $U$ proportional to the radial Green's function for our $D$-dimensional Laplacian.] The corresponding radial equation

$$\left[\frac{d^2}{dr^2}+\frac{D-1}{r}\frac{d}{dr}+2E+\frac{2Z}{r}-\frac{\Lambda(\Lambda+D-2)}{r^2}\right]R(r)=0 \tag{7.20}$$

has solutions regular at the origin which may be written in terms of the confluent hypergeometric function $M(a,b,z)$.[19] Setting

$$E=-\tfrac{1}{2}\kappa^2, \tag{7.21}$$

one finds,

$$R(r)=r^\Lambda\exp(-\kappa r)\,M(\Lambda+\tfrac{1}{2}D-\tfrac{1}{2}-Z/\kappa,\ 2\Lambda+D-1,\ 2\kappa r). \tag{7.22}$$

These radial functions are square-integrable only for discrete values of $\kappa$, which in fact cause $M$ to reduce to a polynomial in $r$. The criterion for this reduction is the following:

$$\begin{aligned} &\Lambda+\tfrac{1}{2}D-\tfrac{1}{2}-Z/\kappa=1+\Lambda-n,\\ &n=\Lambda+1,\ \Lambda+2,\ \Lambda+3,\ldots, \end{aligned} \tag{7.23}$$

which introduces the principal quantum number $n$. Equation (7.23) may be written in terms of $E$ to show the spectrum of bound-state energies,

$$E=-Z^2/2(n+\tfrac{1}{2}D-\tfrac{3}{2})^2. \tag{7.24}$$

It is noteworthy that orbital degeneracy continues to exist for $D\neq 3$.[20] For each $n$, the eigenfunctions with $\Lambda=0,1,\ldots,n-1$ all possess the same energy.

Explicit polynomial forms for the $M$ functions may easily be computed. Some of the simpler cases are now listed.

$$\begin{aligned} &n=1,\ \Lambda=0:\ M(0,D-1,z)=1,\\ &n=2,\ \Lambda=0:\ M(-1,D-1,z)=1-z/(D-1),\\ &n=2,\ \Lambda=1:\ M(0,D+1,z)=1,\\ &n=3,\ \Lambda=0:\ M(-2,D-1,z)=1-\frac{2z}{D-1}+\frac{z^2}{D(D-1)},\\ &n=3,\ \Lambda=1:\ M(-1,D+1,z)=1-\frac{z}{D+1},\\ &n=3,\ \Lambda=2:\ M(0,D+3,z)=1. \end{aligned} \tag{7.25}$$

When $D$ is an integer, the set of solutions $\psi(r,\theta)$, including all possible polar axes, generates the full set of solutions to the spatial wave equation, by taking appropriate linear combinations. Presumably the same is true for noninteger $D$, but a proof is presently lacking. More to the point, it is not yet clear how one can identify a complete orthogonal set of solutions.

## VIII. CLASSICAL PARTITION FUNCTION

Consider $N$ structureless particles of mass $m$, confined to a region $\Omega$ with integer dimension $D$. Let vectors $\mathbf{R}_1\cdots\mathbf{R}_N$ and $\mathbf{P}_1\cdots\mathbf{P}_N$ denote the positions and momenta, respectively, and let $\Phi(\mathbf{R}_1\cdots\mathbf{R}_N)$ be the interaction potential. The classical partition function has the following form:

$$\begin{aligned} Z_N=&(1/N!h^{DN})\int_\Omega d\mathbf{R}_1\cdots\int_\Omega d\mathbf{R}_N\int d\mathbf{P}_1\cdots\int d\mathbf{P}_N\\ &\times\exp[-(\beta/2m)\sum_{j=1}^{N}P_j^2-\beta\Phi(\mathbf{R}_1\cdots\mathbf{R}_N)]. \end{aligned} \tag{8.1}$$

Here $h$ is Planck's constant, and $\beta=1/k_BT$ is the inverse temperature parameter. Contact between $Z_N$ and thermodynamic properties for the system of particles is provided by the Helmholtz free energy $F$,

$$\beta F = -\ln Z_N. \tag{8.2}$$

In the large system limit, with fixed temperature and density $N/\Omega$, the free energy per particle $F/N$ becomes independent of $\Omega$, provided that this region is such that most particles are far from its boundary. In this limit, any convenient shape for $\Omega$ can then be employed, such as the $D$-dimensional "sphere" of appropriate radius.

In seeking to extend $Z_N$ to noninteger $D$, procedures must be indentified for carrying out both momentum and position integrations. The former provide no difficulty, since Axiom A5. immediately affords the result

$$h^{-D} \int d\mathbf{P}_j \exp(-\beta P_j^2/2m) = \lambda_T^{-D}, \tag{8.3}$$

where $\lambda_T$ is the mean thermal deBroglie wavelength,

$$\lambda_T = h/(2\pi m k_B T)^{1/2}. \tag{8.4}$$

However the position integration requires more detailed consideration.

We can use the integration weights $W_n$ introduced by Eq. (3.2), and treat the position integrations as a multiple integral over all distances. The distances involved of course include the $N(N-1)/2$ interparticle separations $r_{ij}$. However we shall in fact treat $\Omega$ as a $D$-dimensional "sphere" (with radius $L$), so that the $N$ distances $r_{0i}$ of the particles from its center are also relevant. Without significant loss of generality, we can suppose that the potential energy $\Phi$ is a function just of the $r_{ij}$.

Under these circumstances, $Z_N$ can be put into the following form:

$$\begin{aligned} Z_n = &(1/N!\lambda_T^{DN}) \int_0^L dr_{01}\, W_1(0|r_{01}) \int_0^L dr_{02} \\ &\times \int_0^{2L} dr_{12}\, W_2(0,1|r_{02}, r_{12}) \int_0^L dr_{03} \int_0^{2L} dr_{13} \\ &\times \int_0^{2L} dr_{23}\, W_3(0,1,2|r_{03}, r_{13}, r_{23}) \cdots \int_0^L dr_{0N} \\ &\times \int_0^{2L} dr_{1N} \int_0^{2L} dr_{N-1,N}\, W_N(0\cdots N|r_{0N}\cdots r_{N-1,N}) \\ &\times \exp[-\beta\Phi(r_{12}\cdots r_{N-1,N})]. \end{aligned} \tag{8.5}$$

Strictly speaking, the upper limits $2L$ on the $r_{ij}$ $(0 < i, j)$ integrals could be extended to infinity, since the affected weights would automatically vanish over the extension.

Evaluation of $Z_N$ in form (8.5) represents no less a formidable challenge than its integer-$D$ predecessor in Eq. (8.1). Nevertheless some of the standard techniques in statistical mechanics can be carried over. In particular it is possible to develop the Ursell—Mayer cluster theory[21] for noninteger $D$. For this purpose we make the conventional simplification that $\Phi$ consists of a sum of central pair potentials,

$$\Phi = \sum_{i<j=1}^{N} \phi(r_{ij}). \tag{8.6}$$

Then the Boltzmann factor $\exp(-\beta\Phi)$ in the partition function may be developed into a sum of products of Mayer $f$ functions,

$$\begin{aligned} \exp(-\beta\Phi) &= \prod_{i<j=1}^{N} [1 + f(r_{ij})] \\ &= 1 + \sum_{i<j=1}^{N} f(r_{ij}) + \sum_{i<j<k=1}^{N} [f(r_{ij})f(r_{jk}) \\ &\quad + f(r_{ij})f(r_{ik}) + f(r_{ik})f(r_{jk}) \\ &\quad + f(r_{ij})f(r_{ik})f(r_{jk})] + \cdots, \end{aligned} \tag{8.7}$$

where

$$f(r_{ij}) = \exp[-\beta\phi(r_{ij})] - 1. \tag{8.8}$$

At this stage one can essentially follow the usual cluster-theory procedure.[21] The only novel feature is the necessity to use contraction properties (3.13) for the weights $W_n$ in the case of integrals containing sets of distances in only a trivial way. Finally one obtains the irreducible cluster expansion for the Helmholtz free energy; in the large-system limit the result has the following form:

$$\frac{\beta F}{N} = \ln\left(\frac{\lambda_T^D N}{e\Omega}\right) - \sum_{k=1}^{\infty} \frac{\beta_k}{k+1}\left(\frac{N}{\Omega}\right)^k. \tag{8.9}$$

The $\beta_k$ are sums of irreducible cluster integrals for $k+1$ particles, and may be expressed thus,

$$\begin{aligned} \beta_k = &(1/k!) \int_0^\infty dr_{12}\, W_1(1|r_{12}) \cdots \int_0^\infty dr_{1,k+1} \cdots \int_0^\infty dr_{k,k+1} \\ &W_k(1\cdots k|r_{1,k+1}\cdots r_{k,k+1})\, S_k(r_{12}\cdots r_{k,k+1}). \end{aligned} \tag{8.10}$$

Here $S_k$ is the sum of those $f$-function products for the $k+1$ particles which correspond to connected graphs without articulation points.

The pressure $p$ for the $N$-particle system may be obtained from $F$ by the relation

$$p = -(\partial F/\ \Omega)_{N,\beta}. \tag{8.11}$$

Within the convergence radius of the cluster expansion (8.9) one therefore has

$$\frac{\beta p \Omega}{N} = 1 - \sum_{k=1}^{\infty} \frac{k\beta_k}{k+1}\left(\frac{N}{\Omega}\right)^k, \tag{8.12}$$

which is the usual virial expansion.

The second virial coefficient in the pressure series (8.12) has the following explicit form:

$$B_2 = -\tfrac{1}{2}\beta_1 = -\tfrac{1}{2}\sigma(D) \int_0^\infty dr\, r^{D-1} f(r). \tag{8.13}$$

If the pair potential $\phi$ describes rigid "spheres" with collision diameters $a$, then

$$\begin{aligned} f(r) &= -1 \quad (0 \le r \le a), \\ &= 0 \qquad (a < r), \end{aligned} \tag{8.14}$$

so that the general-$D$ second virial coefficient becomes

$$B_2 = \frac{\pi^{D/2} a^D}{D\Gamma(D/2)}. \tag{8.15}$$

The third virial coefficient

$$B_3 = -\tfrac{2}{3}\beta_2 \tag{8.16}$$

involves a single cluster integral whose integrand contains the triangular $f$ product, $f(r_{12})f(r_{13})f(r_{23})$. Specifically,

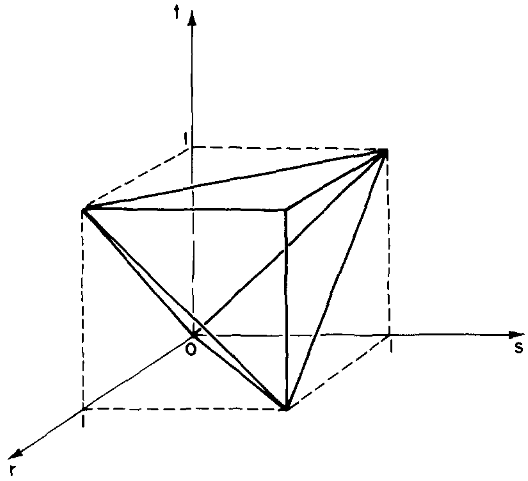


FIG. 3. Hexahedral region (solid lines) over which the generalized third virial coefficient integral (8.19) must be carried out.

$$B_3 = -\tfrac{1}{3}\int_0^\infty dr_{12}\, W_1(1|r_{12}) f(r_{12}) \int_0^\infty dr_{13} \times \int_0^\infty dr_{23}\, W_2(1,2|r_{13},r_{23}) f(r_{13}) f(r_{23}). \quad (8.17)$$

Expressions for $W_1$ and $W_2$ were derived earlier, and allow $B_3$ to be written as

$$B_3 = -\frac{2^{5-D}\pi^{D-1/2}}{3\Gamma(D/2)\Gamma((D-1)/2)} \int_0^\infty dr_{12} \int_0^\infty dr_{13} \times \int_0^\infty dr_{23}\, f(r_{12}) f(r_{13}) f(r_{23})\, r_{12} r_{13} r_{23} \times [2(r_{12}^2 r_{13}^2 + r_{12}^2 r_{23}^2 + r_{13}^2 r_{23}^2) - r_{12}^4 - r_{13}^4 - r_{23}^4]^{(D-3)/2} \times T_0(r_{12}, r_{13}, r_{23}). \quad (8.18)$$

The function $T_0$ is present only to ensure that $r_{12}$, $r_{13}$, and $r_{23}$ can form a triangle, and if they can it is unity; otherwise $T_0$ vanishes. This criterion precisely determines the region over which the quartic factor $[\cdots]$ in Eq. (8.18) is positive.

In the case of rigid "spheres," Eq. (8.18) simplifies somewhat,

$$B_3 = \frac{2^{5-D}\pi^{D-1/2}a^{2D}}{3\Gamma(D/2)\Gamma((D-1)/2)} \times \int_0^1 dr \int_0^1 ds \int_0^1 dt\, rst\, T_0(r,s,t)\, F_D(r,s,t),$$

$$F_D(r,s,t) = [2(r^2s^2 + r^2t^2 + s^2t^2) - r^4 - s^4 - t^4]^{(D-3)/2}. \quad (8.19)$$

Figure 3 shows the region in $r, s, t$ space over which the integral in Eq. (8.19) must be carried out; this hexahedral region is determined both by integration limits and by the condition that $T_0$ be unity. The figure is useful in transforming expression (8.19) to the following form:

$$B_3 = \frac{2^{6-D}\pi^{D-1/2}a^{2D}[I_A(D) + I_B(D)]}{3\Gamma(D/2)\Gamma((D-1)/2)},$$

$$I_A(D) = \int_{1/2}^1 ds \int_{1-s}^s dt \int_{s-t}^1 dr\, rst\, F_D(r,s,t),$$

$$I_B(D) = \int_0^{1/2} dt \int_t^{1-t} ds \int_{s-t}^{s+t} dr\, rst\, F_D(r,s,t) = \frac{\pi^{1/2}\Gamma((D-1)/2)}{4(D-2)^2\Gamma(D/2)} [F(D-2, 2-D; D-1; \tfrac{1}{2}) - 2^{1-D}],$$

where $F(a,b;c;z)$ is the hypergeometric function.[22] Unfortunately $I_A$ does not simplify significantly unless $D$ is an integer. However the form shown is suited for numerical evaluation, should the need arise.

An alternative route to $B_3$ would employ the convolution theorem discussed in Sec. V.

Using the three-center weight in Eq. (4.5), explicit (though complicated) integrals can be worked out for the fourth virial coefficient $B_4 = -\frac{3}{4}\beta_3$.

## IX. DISCUSSION

The preceding exposition implicitly raises a fundamental physical question. Specifically, should we regard the dimension $D$ of the space in which we live as a possibly noninteger quantity that is locally subject to experimental determination? No one can seriously doubt that our world is locally close to three-dimensional. But how close? Results in Sec. IV above show that it does not help much to exhibit three mutually perpendicular lines, since this provides neither a necessary nor a sufficient condition for $D$ to equal 3.

Probably the most direct experimental approach to determination of $D$ would be the measurement of mass content of a series of homogeneous spherical bodies. The expected result for $D=3$ of course is that this mass would be strictly proportional to the radius (or diameter) cubed. However, accumulated errors in weighing, in size and shape measurement, and in density variations (due to composition and temperature inhomogeneity, and to body stresses) would likely limit the precision in determination of the exponent $D$ to about 1 part in $10^6$. By this means one presumably would conclude that $D$ was $3 \pm 10^{-6}$ in our terrestrial locale.

In seeking alternative procedures with greater precision, it might be valuable to examine mathematically how spheres pack when $D$ departs slightly from 3. With $D=3$ exactly, spheres can be fitted together in infinitely extended close packings (f.c.c., h.c.p., or hybrids of the two) with each sphere touching twelve neighbors. If $D$ were slightly larger than 3, attempts to build a known $D=3$ packing outward from a central sphere would begin to produce gaps, eventually allowing extra sphere insertions. By contrast, the case with $D$ slightly less than three would not permit a full complement of spheres to pack properly in the successive shells expected for $D=3$; in terms of material spheres forced into those shells, an accumulation of elastic stress would result. Proper interpretation of the physical construction of large sphere packings thus might help to place tight bounds on our ambient dimension.

In any case, experiments designed to determine $D$ to 1 part in $10^9$ or better would likely require the utmost sophistication in concept and perserverence in execution.

In general relativity, gravitational fields are understood to be geometric perturbations (curvatures) in our spacetime,[23] rather than entities residing within a flat spacetime. The concept that physical force fields generally might be related to purely geometric distortions in space is appealing, and leads one to inquire if dimension $D$ itself might not play an important role as a field variable. The preceding development has considered only uniform spaces $\mathcal{S}_D$ for which $D$ had a fixed value. However a more general class of spaces can also be generated within which $D$ varies continuously from point to point (integration weights $W_n$ would exhibit the change explicitly). Under the assumption that general relativity is an incomplete description of reality, it might be appropriate to ask if regions of strong gravitational field display perturbed dimension. More generally, local space dimension may provide geometric field variables in addition to those of general relativity, that would have a place in a unified description of all the forces in nature.

Finally, mention should be made of a paper by Wilson,[24] which also offers an axiomatic description of spaces with noninteger dimension. While most of Wilson's results on integrals appear to be consistent with those deduced here, it is not at all clear that the mathematical spaces generated in the two approaches are isomorphic. In particular, Wilson permits vector addition, and requires an infinite number of vector components when $D$ is not an integer; in the present case vector addition [Eq. (2.10)] has explicitly been excluded, and we have seen that negative integration weights inevitably occur.

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